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LETTER TO THE EDITOR

Discussion of "Torsional vibrations of a circular disk on an infinite transversely isotropic medium", *Int. J. Solids Structures*, Vol. 25, No. 9, pp. 1069–1076 (1989)

In a recent article, Tsai (1989) presented an analytical formulation for the mixed boundary value problem related to time-harmonic torsional vibrations of a circular rigid disk bonded to the surface of a transversely isotropic elastic half space. Numerical solutions for the compliance function, contact shear stress and resonant amplitudes were also presented. The study by Tsai (1989) is a welcome addition to the literature on contact problems and the numerical results are useful to engineering practice. A review of the solution procedure, however, indicates that the solution for the problem can be obtained from the solution for the isotropic case (Reissner and Sagoci, 1944; Gladwell, 1969; Keer *et al.*, 1974; Luco, 1976) through the use of a scalar transformation. Therefore, a complete analysis is unnecessary. The following presents the proof of this statement and it is shown that similar correspondence exists, even in the case of a transversely isotropic elastic layer.

Consider the axisymmetric torsional vibration of a transversely isotropic elastic half space. A cylindrical coordinate system (r, θ, z) is employed in the analysis with the z-axis parallel to the material axis of symmetry. The entire problem, including the coordinate frame, is nondimensionalized with respect to the radius a of the disk. The nonzero stress components $\sigma_{r\theta}$ and $\sigma_{\theta z}$ can be expressed in terms of the displacement v in the θ-direction, as:

$$\sigma_{r\theta} = \frac{(c_{11}-c_{12})}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \tag{1a}$$

$$\sigma_{z\theta} = c_{44}\frac{\partial v}{\partial z}, \tag{1b}$$

where c_{11}, c_{12} and c_{44} are material constants (Lekhnitskii, 1963).

The displacement v is governed by

$$\frac{(c_{11}-c_{12})}{2}\left[\frac{\partial^2 v}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right] + c_{44}\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2}, \tag{2}$$

where ρ is the density of the medium.

It is assumed that the motion is time-harmonic with circular frequency ω. A solution of eqn (2) can be expressed as:

$$v(r,z) = \int_0^\infty [A(\xi)\,\mathrm{e}^{-\beta\bar{z}} + B(\xi)\,\mathrm{e}^{\beta\bar{z}}] J_1(\xi r)\,\mathrm{d}\xi, \tag{3}$$

where

$$\beta^2 = \xi^2 - k^2; \quad \bar{z} = z/\gamma; \quad \gamma = \left(\frac{2c_{44}}{c_{11}-c_{12}}\right)^{1/2}; \quad k^2 = \frac{\rho\omega^2\gamma^2}{c_{44}}$$

and J_1 is the Bessel function of the first kind and first order.

In view of eqns (1) and (3), the shear stress $\sigma_{z\theta}$ can be expressed as:

$$\sigma_{z\theta}(r,z) = \frac{c_{44}}{\gamma}\int_0^\infty \beta[-A(\xi)\,e^{-\beta z}+B(\xi)\,e^{\beta z}]\,J_1(\xi r)\,d\xi. \tag{4}$$

The boundary value problem related to torsional vibration of the rigid disk can be expressed as:

$$v(r,0) = \phi_0 r, \qquad 0 \leqslant r < 1 \tag{5a}$$

$$\sigma_{z\theta}(r,0) = 0, \qquad r > 1 \tag{5b}$$

where ϕ_0 is the rotation of the disk about the z-axis.

Substitution of eqns (3) and (4) into eqn (5), together with the condition that $B(\xi) \equiv 0$ for an elastic half space region ($0 \leqslant z < \infty$), yields the following dual integral equation system for $A(\xi)$:

$$\int_0^\infty A(\xi)J_1(\xi r)\,d\xi = \phi_0 r \qquad 0 \leqslant r < 1; \tag{6a}$$

$$\int_0^\infty \beta A(\xi)J_1(\xi r)\,d\xi = 0 \qquad r > 1. \tag{6b}$$

The torque T_0 acting on the disk can be expressed as:

$$T_0 = -\frac{2\pi c_{44}a^3}{\gamma}\int_0^1 r^2\left[\int_0^\infty \beta A(\xi)J_1(\xi r)\,d\xi\right]dr. \tag{7}$$

Now, consider an identical boundary value problem for an isotropic half space with shear modulus c_{44} and mass density ρ. Let $\bar{\omega}$ denote the circular frequency in this case; the corresponding general solution for displacement $\bar{v}$ can be expressed as:

$$\bar{v}(r,z) = \int_0^\infty [\bar{A}(\xi)\,e^{-\bar{\beta} z}+\bar{B}(\xi)\,e^{\bar{\beta} z}]J_1(\xi r)\,d\xi, \tag{8}$$

where

$$\bar{\beta}^2 = \xi^2-\bar{k}^2 \quad\text{and}\quad \bar{k}^2 = \frac{\rho\bar{\omega}^2}{c_{44}}.$$

The boundary value problem is governed by

$$\int_0^\infty \bar{A}(\xi)J_1(\xi r)\,d\xi = \phi_0 r, \qquad 0 \leqslant r < 1; \tag{9a}$$

$$\int_0^\infty \bar{\beta}\bar{A}(\xi)J_1(\xi r)\,d\xi = 0, \qquad r > 1. \tag{9b}$$

The torque $\bar{T}_0$ acting on the disk can be expressed as:

$$\bar{T}_0 = -2\pi c_{44} a^3 \int_0^1 r^2 \left[\int_0^\infty \bar{\beta}\bar{A}(\xi) J_1(\xi r)\, d\xi \right] dr. \tag{10}$$

It is noted that if $\bar{k} = k$ then $\beta = \bar{\beta}$ and comparison of eqns (6) and (9) yields $\bar{A}(\xi) = A(\xi)$. In addition, comparison of eqns (7) and (10) yields $T_0 = \bar{T}_0/\gamma$. The condition $\bar{k} = k$ implies that $\bar{\omega} = \omega\gamma$. Therefore, the torque T_0 at frequency ω corresponding to the transversely isotropic problem is equal to $1/\gamma$ times the torque corresponding to the isotropic problem at frequency $\omega\gamma$. An identical relation exists between the shear stress $\sigma_{\theta z}$.

Next, consider the case of a rigid circular disk bonded to a transversely isotropic elastic layer of nondimensional thickness h, overlying a rigid base. In this case, $B(\xi) \neq 0$ in eqn (3). The boundary value problem is defined by eqns (5) together with the condition

$$v(r, h) = 0 \tag{11}$$

Equation (11) implies that

$$B(\xi) = -A(\xi)\, e^{-2\beta h^*}, \tag{12}$$

where $h^* = h/\gamma$.
The boundary value problem is governed by

$$\int_0^\infty A(\xi)\,[1 - e^{-2\beta h^*}] J_1(\xi r)\, d\xi = \phi_0 r, \qquad 0 \leqslant r < 1; \tag{13a}$$

$$\int_0^\infty \beta A(\xi)[-1 + e^{-2\beta h^*}] J_1(\xi r)\, d\xi = 0, \qquad r > 1. \tag{13b}$$

An identical problem for an isotropic layer of nondimensional thickness $\bar{h}$ is governed by

$$\int_0^\infty \bar{A}(\xi)[1 - e^{-2\bar{\beta}\bar{h}}] J_1(\xi r)\, d\xi = \phi_0 r \tag{14a}$$

$$\int_0^\infty \bar{\beta}\bar{A}(\xi)[-1 + e^{-2\bar{\beta}\bar{h}}] J_1(\xi r)\, d\xi = 0. \tag{14b}$$

It is evident from eqns (13) and (14) that if $\beta = \bar{\beta}$ and $h^* = \bar{h}$, then $A(\xi) = \bar{A}(\xi)$.

Therefore, the torque corresponding to a transversely isotropic layer of height h at frequency ω is equal to $1/\gamma$ times the torque corresponding to an isotropic layer of thickness h/γ at frequency $\omega\gamma$. A similar correspondence exists between the stress $\sigma_{z\theta}$. Numerical solutions for the isotropic elastic layer problem are given by Gladwell (1969) and Keer *et al.* (1974).

It is noted that the above type of direct correspondence between isotropic and transversely isotropic material exists only for a *limited class of problems involving axisymmetric torsional deformations*. In the case of general deformations, a formal solution is required. A recent article by Wang and Rajapakse (1990) considers general asymmetric boundary value problems related to a transversely isotropic medium under static loading, and elastodynamic problems (Rajapakse and Wang, 1991) are also currently under study.

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REFERENCES

Gladwell, G. M. L. (1969). The forced torsional vibration of an elastic stratum. *Int. J. Engng Sci.* **7,** 1011–1024.

Keer, L. M., Jabali, H. H. and Chantaamungkorn, K. (1974). Torsional oscillations of a layer bonded to an elastic half space. *Int. J. Solids Structures* **10,** 1–13.

Lekhnitskii, S. G. (1963). *Theory of Anisotropic Elastic Bodies*. Holden-Day, San Francisco, CA.

Luco, J. E. (1976). Torsional response of structures for SH waves: The case of hemispherical foundation. *Bull. Seismol. Soc. Am.* **66,** 109–123.

Rajapakse, R. K. N. D. and Wang, Y. (1991). Elastodynamic Green's functions of an orthotropic half plane. *J. Engng Mech., ASCE* (in press).

Reissner, E. and Sagoci, H. F. (1944). Forced torsional oscillations of an elastic half space. *J. Appl. Phys.* **15,** 652–654.

Tsai, Y. M. (1989). Torsional vibrations of a circular disk on an infinite transversely isotropic medium. *Int. J. Solids Structures* **25**(9), 1069–1076.

Wang, Y. and Rajapakse, R.K.N.D. (1990). Asymmetric boundary-value problems for a transversely isotropic elastic medium. *Int. J. Solids Structures* **26**(8), 833–849.